Conjugate Duality of Set–Valued Functions

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Abstract

To a function with values in the power set of a pre–ordered, separated locally convex space a family of scalarizations is given which completely characterizes the original function. A concept of a Legendre–Fenchel conjugate for set-valued functions is introduced and identified with the conjugates of the scalarizations. The concept of conjugation is connected to the notion of $(*, s)$–dualities and duality results are provided. **keywords:** conjugate duality; set–valued function; conlinear space; residuation; set relations **Clesscode:** 49N15; 90C46; 90C48

1 Introduction

In this paper, we will introduce a duality theory for set–valued functions. We will apply our theory proving a biconjugation theorem, a sum– and a chain–rule and weak and strong duality results of Fenchel–Rockafellar type. Apart from the purely academic interest, investigating set–valued functions is motivated by their applicability in financial mathematics, compare [5, 17, 22], and in vector optimization.

The idea in vector optimization is to present an optimal vector, solving a minimization problem on a vector space. The common lack of infima and suprema in vector spaces makes it obvious, that 'vectorial' constructions are in general not appropriate to solve these problems, compare [14]. Even if investigations are restricted to order complete, partially ordered spaces or subsets, least upper or greatest lower bounds can be 'far away' and have little in common with the function in question.

An approach to solve this dilemma is to search for a set of effective elements and hereby transform the original vector–valued problem into a set–valued problem.

Apparently, as pointed out by J.Jahn, 'the best football team is not necessarily the team with the best player', thus we understand a optimal solution of a set–valued minimization problem to be a set rather than a single point. Consequently we investigate on set–valued functions, understanding the images to be elements of the power set of a vector space.

The basic idea is to extend the order on the space Z, given by a convex cone $C \subseteq Z$ to an order on $\mathcal{P}(Z)$ in an appropriate way, compare [13, 16, 21] and identify a function $f : X \to \mathcal{P}(Z)$ with its epigraphical extension $f_C(x) = f(x) + C$. The same extension can of course be done for vector–valued functions, thus as a special amenity our theory includes vector–valued functions as special class of set–valued functions. We identify the

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subset $\mathcal{P}^{\triangle} \subseteq \mathcal{P}(Z)$ as the set of all elements $A = A + C \in \mathcal{P}(Z)$. This set is an order complete lattice w.r.t. $\supseteq$ and contains all images of functions of epigraphical type. Equipped with an appropriate addition and multiplication with nonnegative reals, $\mathcal{P}^{\triangle}$ is a so–called inf–residuated conlinear space, compare [16, 18], thus supplies sufficient algebraic structure to introduce coherent definitions of conjugates, directional derivatives and subdifferentials. At present, we will concentrate on the investigation of conjugates, more can be found in [29], compare also [8] and especially [25] for closely related approaches.

The introduced notion of conjugate functions turns out to be an example of a $(*, s)$–duality, see [11] and many properties from the well known scalar conjugate can be rediscovered in our setting.

In [14, 15], duality results are proven directly, using separation arguments in the space $X \times Z$. In contrast, we introduce a family of scalar functions related to and completely characterizing a set–valued function. The set–valued conjugate is also determined by the family of conjugates of the scalarizations of the original function.

Because of the one-to-one correspondence between set–valued functions and their conjugates on the one hand, the family of scalarizations and their conjugates on the other, we can utilize results from the scalar theory to obtain results for the set–valued case.

The article is organized as follows. In Section 2, we set up the framework for the investigations presented in this paper by recalling some notions on dualities and conjugation in Subsection 2.1 and the definition of a residuated conlinear space in Subsection 2.2. In Section 3, a theory for extended real valued functions is introduced. This will be used later on, as the scalarizing functions introduced in Section 5 are in general not proper. In Section 4, the class of epigraphical type functions from a locally convex space to the power set of another locally convex space is introduced. The image space of these functions is identified as a subset of the original power set. This subset has sufficient algebraic, order theoretic and topological structures to establish set–valued analogs for all major results from the scalar theory. The definitions of the conjugate and biconjugate are provided in Section 6, while a selection of duality results is presented in Section 7.

2 General concepts and notations

2.1 Conjugation and dualities

We recall that for a locally convex space X with topological dual X^*, the Fenchel conjugate of a function $f : X \to \overline{\mathrm{IR}}$ is $f^* : X^* \to \overline{\mathrm{IR}}$ defined by

$$\forall x^* \in X^* : \quad f^*(x^*) = \sup_{x \in X} \left(\phi(x^*, x) - f(x)\right), \tag{2.1}$$

with $\phi(x^*, x) = x^*(x)$ for all $x^* \in X^*$, $x \in X$ and setting $r - (\pm\infty) = \mp\infty$ for $r \in \mathrm{IR}$. This concept has been extended by Moreau [26, 14.(c)], [27, 4.(c)], replacing X and X^* by arbitrary sets V and W and ϕ by a so called coupling function $\phi : V \times W \to \overline{\mathrm{IR}}$. Moreover, the difference operator in Equation (2.1) has been replaced by an operation on $\overline{\mathrm{IR}}$, which is defined by

$$r \dotplus - s = \begin{cases} -\infty, & \text{if } r = -\infty \text{ or if } s = +\infty; \\ r - s, & \text{if } r, s \in \mathrm{IR}; \\ +\infty, & \text{else.} \end{cases}$$

The so called Moreau–Fenchel conjugate of $f : V \to \overline{\mathrm{I\!R}}$ associated to the coupling function ϕ is $f^{\phi} : W \to \overline{\mathrm{I\!R}}$, defined by

$$\forall w \in W : \quad f^{\phi}(w) = \sup_{v\in V} \left(\phi(w,v) \dot{+} - f(x)\right). \tag{2.2}$$

In [11], the concept of $(*, s)$-dualities is introduced as a generalization of (2.1). The set $\overline{\mathrm{I\!R}}$ is replaced by a complete lattice $Z = (Z, \leq)$ and the addition by a binary (commutative) operation $* : Z \times Z \to Z$ with the property

$$\forall M \subseteq Z,\, z \in Z : \quad z * \inf M = \inf_{m\in M} (z * m). \tag{2.3}$$

By Property (2.3), $+\infty := \inf \emptyset$ dominates the operation $*$ and the inf–difference or inf–residuation of z_1 and z_2 is defined by

$$z_1 *_u^{-1} z_2 = \inf \{t \in Z : \; z_1 \leq z_2 * t\} \tag{2.4}$$

and $z_1 \leq z_2 * \left(z_1 *_u^{-1} z_2\right)$. To a bijection $s : Z \to Z$, the operation $*^s : Z \times Z \to Z$ is defined as

$$z_1 *^s z_2 = s(s^{-1}(z_1) * z_2)$$

for all $z_1, z_2 \in Z$.

The sets $Z^V = \{f : V \to Z\}$ and $Z^W = \{f : W \to Z\}$ are ordered by the point–wise ordering and $f_1 * f_2$ and $s(f)$ are understood as point–wise operations.

A function $\mathrm{c} : Z^V \to Z^W$ with $\mathrm{c}\,(f) = f^{\mathrm{c}}$ is called a $(*, s)$–duality, if and only if it satisfies the following two conditions

$$\forall \{f_i\}_{i\in I} \subseteq Z^V : \quad \left(\inf_{i\in I} f_i\right)^{\mathrm{c}} = \sup_{i\in I} f_i^{\mathrm{c}}; \tag{2.5}$$

$$\forall f \in Z^V,\, \forall z \in Z : \quad (f * z)^{\mathrm{c}} = f^{\mathrm{c}} *^s z. \tag{2.6}$$

The dual of $\mathrm{c} : Z^V \to Z^W$ is the mapping $\mathrm{c}' : Z^W \to Z^V$, given by

$$\forall v \in V : \quad f^{\mathrm{c}'}(v) = \inf \left\{g \in Z^V \mid g^{\mathrm{c}} \leq f\right\}. \tag{2.7}$$

If $*$ has a neutral element, then f^{c} can be expressed via a coupling function $\phi : V \times W \to Z$,

$$f^{\mathrm{c}}(w) = \sup_{v\in V} (\phi(x,w) *^s f(x)) \tag{2.8}$$

for all $f \in Z^V$ and all $w \in W$.

If additionally s is a duality and $(*_u^{-1})^s$ is commutative and associative, then the dual of f^{c} admits the representation

$$\forall v \in V : \quad f^{\mathrm{c}'}(v) = \sup_{w\in W} \left(\phi(v,w) *_u^{-1} f^{\mathrm{c}}(v)\right). \tag{2.9}$$

It turns out that the set–valued conjugate introduced in course of this paper admits the representation as in Formula (2.8), the biconjugate is of the form presented in Formula (2.9).

2.2 Conlinear spaces

In [16, Section 2.1.2], the concept of a conlinear space has been introduced. A set $Z = (Z, +, \cdot)$ is called a conlinear space, if $(Z, +)$ is a commutative monoid with neutral element θ and for all $z, z_1, z_2 \in Z$, $r, s \in \mathbb{R}_+$ holds $r(z_1 + z_2) = rz_1 + rz_2$, $r(sz) = (rs)z$ and $1z = z$, $0z = \theta$.

If $(Z, \leq)$ is a complete lattice and $\leq$ is compatible with the algebraic structure in $(Z, +, \cdot)$, then $(Z, +, \cdot, \leq)$ is called a order complete conlinear space.

If additionally the operation $+ : Z \times Z \to Z$ satisfies Property (2.3), then the inf–residuation $\dot{-} : Z \times Z \to Z$ defined by

$$z_1 \dot{-} z_2 = z_1 \overset{-1}{+_u} z_2 = \inf \{t \in Z \mid z_1 \leq z_2 + t\}$$

for all $z_1, z_2 \in Z$ replaces the usual difference operation, $z_1 \leq z_2 + (z_1 \dot{-} z_2)$ and $Z = (Z, +, \cdot, \leq)$ is an called an inf–residuated order complete conlinear space.

Residuation seems to be introduced by Dedekind, [7, p. 71],[6, p.329-330], compare also [2, XIV, §5], [9, Chap. XII] or [10, Chap. 3] on sup–residuation. In [10, Lemma 3.3] the concept of inf–residuation is indicated. The corresponding residuations may substitute the difference operation on $\mathbb{R}$.

A conlinear inf–residuated order complete space as image space supplies sufficient structure to apply the concepts introduced in [11] and to define convexity of functions, as multiplication with positive reals is defined as was done in [14]. Multiplication with -1 leads to a sup–residuated conlinear space, see [16] for details.

Also, the structure carries over from a space Z to the set of all subsets, $\mathcal{P}(Z)$, provided the algebraic and order relations are extended to relations on $\mathcal{P}(Z)$ in an appropriate way, see [16, Theorem 13] and [18] for more details.

3 Extended real–valued functions

In this chapter, we briefly recall some facts on extended-real valued functions. For a more thorough investigation the reader is referred to [18] and the references therein.

3.1 The extended real numbers

The set $\mathbb{R}$ is extended to $\overline{\mathbb{R}}$ in the usual way by adding two elements $+\infty$ and $-\infty$. Setting

$$\begin{aligned} &\forall t \in \mathbb{R} : \quad -\infty \leq t \leq +\infty; \\ &\inf \emptyset = +\infty, \quad \sup \emptyset = -\infty, \end{aligned}$$

the set $(\overline{\mathbb{R}}, \leq)$ is an order complete lattice.

The addition $+ : \mathbb{R} \times R \to \mathbb{R}$ on $\mathbb{R}$ admits two distinct extensions to an operator on $\overline{\mathbb{R}}$, the inf– and sup–addition.

$$r \dot{+} s = \inf \{a + b \mid a, b \in \mathbb{R},\ r \leq a,\ s \leq b\}; \tag{3.1}$$

$$r \underset{\cdot}{+} s = \sup \{a + b \mid a, b \in \mathbb{R},\ a \leq r,\ b \leq s\} \tag{3.2}$$

for all $r, s \in \overline{\mathbb{R}}$. Both inf– and sup–addition are commutative and compatible with the usual order $\leq$ on $\overline{\mathbb{R}}$, that is if $t_1 \leq t_2$ for $t_1, t_2 \in \overline{\mathbb{R}}$, then for all $s \in \overline{\mathbb{R}}$ it holds $s \dot{+} t_1 \leq s \dot{+} t_2$ and

$s \underset{\cdot}{+} t_1 \le s \underset{\cdot}{+} t_2$. The greatest element $+\infty$ dominates the inf–addition $\dot{+}$, $-\infty$ dominates the sup–addition *ssum*.

$$\forall r \in \overline{\mathrm{IR}}: \quad (+\infty) \dot{+} r = +\infty, \ (-\infty) \underset{\cdot}{+} r = -\infty. \tag{3.3}$$

The construction in Equation (3.1) has been introduced in [6, p.329-330], in order to extend the addition from the set of rational to the set of real numbers. Also, it is indicated that other operations, especially the difference, can be introduced equivalently, which indicates the concept of inf– and sup–difference, see Definition 3.2.

Proposition 3.1 *[18] Let $M \subseteq \overline{\mathrm{IR}}$ and $r \in \overline{\mathrm{IR}}$, then*

$$\begin{aligned} \inf\left(\{r\} \dot{+} M\right) &= r \dot{+} \inf M; \\ \sup\left(\{r\} \dot{+} M\right) &\le r \dot{+} \sup M; \\ \inf\left(\{r\} \underset{\cdot}{+} M\right) &\le r \underset{\cdot}{+} \sup M; \\ \sup\left(\{r\} \underset{\cdot}{+} M\right) &= r \underset{\cdot}{+} \inf M. \end{aligned}$$

understanding the sum of sets to be the Minkowsky sum.

Proposition 3.1 tells us that $\left(\overline{\mathrm{IR}}, \dot{+}, \le\right)$ and $\left(\overline{\mathrm{IR}}, \underset{\cdot}{+}, \le\right)$ are inf– and sup–residuated lattices.

Definition 3.2 *Let $r, s \in \overline{\mathrm{IR}}$. The* inf– *and* sup–*residual of r and s are defined as*

$$r \dot{-} s = \inf\left(t \in \overline{\mathrm{IR}} \mid r \le s \dot{+} t\right) \tag{3.4}$$

$$r \underset{\cdot}{-} s = \sup\left(t \in \overline{\mathrm{IR}} \mid s \underset{\cdot}{+} t \le r\right). \tag{3.5}$$

Again, $r \le s \dot{+} (r \dot{-} s)$ and $s \underset{\cdot}{+} (r \underset{\cdot}{-} s) \le r$ holds true.

Multiplication with nonnegative reals is extended to $\overline{\mathrm{IR}}$ by setting $t \cdot (\pm\infty) = \pm\infty$, if $t > 0$ and $0 \cdot (\pm\infty) = 0$. The set $\overline{\mathrm{IR}}$ supplied with either addition, multiplication with nonnegative reals and the extended order relation is a order complete residuated conlinear space. We define

$$\mathrm{IR}^{\triangle} = (\overline{\mathrm{IR}}, \dot{+}, \cdot, \le); \quad \mathrm{IR}^{\triangledown} = (\overline{\mathrm{IR}}, \underset{\cdot}{+}, \cdot, \le).$$

Moreover, the multiplication with -1 is defined as $(-1) \cdot (\pm\infty) = \mp\infty$. We abbreviate $(-1)r = -r$, when no confusion can arise. Obviously, for $M \subseteq \overline{\mathrm{IR}}$ and $r, s \in \overline{\mathrm{IR}}$ the following is satisfied.

Proposition 3.3 *[18] Let $M \subseteq \overline{\mathrm{IR}}$ and $r, s \in \overline{\mathrm{IR}}$, then*

$$\begin{aligned} &(-1) \inf M = \sup (-1) M; \\ &r \dot{+} (-s) = r \underset{\cdot}{-} s, \quad -(r \dot{+} s) = (-s) \underset{\cdot}{+} (-r); \\ &r \underset{\cdot}{+} (-s) = r \dot{-} s, \quad -(r \underset{\cdot}{+} s) = (-s) \dot{+} (-r). \end{aligned}$$

Multiplication with -1 is a duality between $\mathrm{IR}^{\triangle}$ and $\mathrm{IR}^{\triangledown}$.

3.2 Extended real–valued functions

Let X be a topological linear space and $g : X \to \mathbb{R}^{\vartriangle}$ a function. Multiplication with -1 transfers g to $-g : X \to \mathbb{R}^{\triangledown}$. We will concentrate on the first type of functions in the sequel, keeping in mind that for the second class symmetric results can be proven. To $g : X \to \mathbb{R}^{\vartriangle}$, the epigraph and effective domain of g are defined, as usual,

$$\begin{aligned} \operatorname{epi} g &= \{(x,r) \in X \times \mathbb{R} \mid g(x) \le r\}\,, \\ \operatorname{dom} g &= \{x \in X \mid g(x) \ne +\infty\}\,. \end{aligned}$$

We say $g \le f$, iff $\operatorname{epi} g \supseteq \operatorname{epi} f$.

The set $(\mathbb{R}^{\vartriangle})^X = \{g : X \to \mathbb{R}^{\vartriangle}\}$ equipped with the point–wise addition, multiplication with nonnegative reals and order relation is a order complete, inf–residuated conlinear space.

A function $g : X \to \mathbb{R}^{\vartriangle}$ is said to be proper, if $\operatorname{dom} g \ne \emptyset$ and for all $x \in X$ exists a $t \in \mathbb{R}$ such that $(x,t) \notin \operatorname{epi} g$. It is said to be closed (convex), if its epigraph is a closed (convex) set. It is subadditive, if $\operatorname{epi} g$ is closed under addition, i.e. $\operatorname{epi} g \dotplus \operatorname{epi} g \subseteq \operatorname{epi} g$ and positively homogeneous, if $\operatorname{epi} g$ is a cone, i.e.

$$\operatorname{epi} g = \{t(x,r) \in X \times \mathbb{R} \mid t > 0, (x,r) \in \operatorname{epi} g\}\,.$$

The conical hull of g is given by $\operatorname{epi}(\operatorname{cone} g) = \operatorname{cone} \operatorname{epi} g$ with

$$\operatorname{cone} \operatorname{epi} g = \{t(x,r) \in X \times \mathbb{R} \mid t > 0, (x,r) \in \operatorname{epi} g\}\,.$$

For reasons that will be illustrated in Example 3.4, we do not presume $(0,0) \in \operatorname{cone} \operatorname{epi} g$. A positively homogeneous convex function is called sublinear.

It does not make much sense to define concavity or superlinearity for $g : X \to \mathbb{R}^{\vartriangle}$, as for these concepts $\mathbb{R}^{\triangledown}$ is the appropriate image space.

Example 3.4 (Improper affine functions on linear spaces)*[18] Let X be a topological linear space and X^* its topological dual. We write $x^*(x)$ for the value of an element $x^* \in X^*$ at $x \in X$. Let $r \in \mathbb{R}$ and set $x^*_r(x) = x^*(x) - r$ for all $x \in X$. Each $x^* \in X^*$ generates a closed improper* inf*–extension of the affine function $x \mapsto x^*_r(x)$ by*

$$\hat{x}^*_r(x) = \begin{cases} -\infty & : \quad x^*_r(x) \le 0 \\ +\infty & : \quad x^*_r(x) > 0 \end{cases}$$

If $x^* = 0$, then we obtain $\hat{x}^*_r = -\infty$ if $r \le 0$, and $+\infty$ else. Analogously, the improper sup-extension of $x \mapsto x^*_r(x)$ (with a closed hypograph) can be obtained by reversing the roles of $-\infty$ and $+\infty$.

We write x^* for x^*_0 and $\hat{x}^*$ for $\hat{x}^*_0$ and define

$$\hat{X}^* = \{\hat{x}^* \mid x^* \in X^*\}$$

and $X^{\vartriangle} = X^* \cup \hat{X}^*$, the (topological) inf–dual of X. Note that the functions $\hat{x}^*$ are subadditive mappings into $\mathbb{R}^{\vartriangle}$ and superadditive mappings into $\mathbb{R}^{\triangledown}$, but not additive, i.e. in general $\hat{x}^*(x_1 + x_2) \ne \hat{x}^*(x_1) \dotplus \hat{x}^*(x_2)$. However, in [18] it is proven that

$$\begin{aligned} \xi_r(x_1 + x_2) &= \sup_{r_1 + r_2 = r} \xi_{r_1}(x_1) \underset{\cdot}{+} \xi_{r_2}(x_2) \\ \xi_r(x_1 - x_2) &= \sup_{r_1 + r_2 = r} \xi_{r_1}(x_1) \stackrel{\cdot}{-} \xi_{-r_2}(x_2) \end{aligned}$$

applies for all affine functions $\xi \in X^{\triangle}$, $r \in \mathbb{R}$ and all $x_1, x_2 \in X$.

Defining the closed convex hull $(\operatorname{cl}\operatorname{co} g) : X \to \mathbb{R}^{\triangle}$ of a function g via $\operatorname{epi}(\operatorname{cl}\operatorname{co} g) = \operatorname{cl}\operatorname{co}(\operatorname{epi} g)$, it is well known that $\operatorname{cl}\operatorname{co} g$ is the point–wise supremum of the proper closed affine minorants of $g : X \to \mathbb{R}^{\triangle}$ if and only if $\operatorname{cl}\operatorname{co} g$ is proper or constant $+\infty$ or $-\infty$. A improper affine function $\hat{x}^*_r$ is a minorant of $g : X \to \mathbb{R}^{\triangle}$, if and only if x^*_r is a minorant of $I_{\operatorname{dom} g}$, the indicator function of $\operatorname{dom} g$ defined by $I_{\operatorname{dom} g}(x) = 0$, if $x \in \operatorname{dom} g$ and $I_{\operatorname{dom} g}(x) = +\infty$, else:

$$\hat{x}^*_r(x) \leq g(x) \quad \Leftrightarrow \quad x^*(x) - r \leq I_{\operatorname{dom} g}(x). \tag{3.6}$$

If $(\operatorname{cl}\operatorname{co} g) : X \to \mathbb{R}^{\triangle}$ is improper, then $(\operatorname{cl}\operatorname{co} g)(x) \in \{\pm\infty\}$ for all $x \in X$ and $\operatorname{dom}(\operatorname{cl}\operatorname{co} g)$ is a closed convex set. In this case, $\operatorname{cl}\operatorname{co} g$ is the point–wise supremum of its improper affine minorants.

$$\forall x \in X : \quad (\operatorname{cl}\operatorname{co} g)(x) = \sup_{\substack{\hat{0}_r \leq g;\\ r \in \mathbb{R}}} \hat{0}_r(x). \tag{3.7}$$

Thus, for any function $g : X \to \mathbb{R}^{\triangle}$ the following equation is satisfied.

$$\forall x \in X : \quad (\operatorname{cl}\operatorname{co} g)(x) = \sup_{\substack{\xi_r \leq g;\\ (\xi, r) \in (X^{\triangle} \times \mathbb{R})}} \xi_r(x). \tag{3.8}$$

Definition 3.5 *The conjugate of $g : X \to \mathbb{R}^{\triangle}$ is the function $g^* : X^* \to \mathbb{R}^{\triangle}$ given by*

$$\forall x^* \in X^* : \quad g^*(x^*) = \sup_{x \in X} \left(x^*(x) \dot{-} g(x) \right), \tag{3.9}$$

the biconjugate $g^{**} : X \to \mathbb{R}^{\triangle}$ is given by

$$\forall x \in X : \quad g^{**}(x) = \sup_{x^* \in X^*} \left(x^*(x) \dot{-} g^*(x) \right). \tag{3.10}$$

In [18, Definition 5.8], a conjugate function with dual set $X^{\triangle} \times \mathbb{R}$, the set of affine functions is defined by

$$\forall (\xi, r) \in X^{\triangle} \times \mathbb{R} : \quad g^{\mathrm{c}}(\xi, r) = \sup_{x \in X} \left\{ \xi_r(x) \dot{-} g(x) \right\}.$$

If $\xi = x^* \in X^*$, then $g^{\mathrm{c}}(x^*, r) = g^*(x^*) \dot{-} r$. Otherwise,

$$g^{\mathrm{c}}(\hat{x}^*, r) = \begin{cases} -\infty, & \text{if } \hat{x}^*_r \text{ is a minorant of } g; \\ +\infty, & \text{else} \end{cases}$$

for all $(x^*, r) \in X^* \times \mathbb{R}$. Applying Equation (3.6), for all $x^* \in X^*$, $r \in \mathbb{R}$ we obtain

$$g^{\mathrm{c}}(\hat{x}^*, r) = -\infty \quad \Leftrightarrow \quad (I_{\operatorname{dom} g})^{\mathrm{c}}(x^*_r) = (I_{\operatorname{dom} g})^*(x^*) \dot{-} r \leq 0.$$

The mapping $\mathrm{c} : (\mathbb{R}^{\triangle})^X \to (\mathbb{R}^{\triangle})^{X^*}$ with $\mathrm{c}(g) = g^*$ is a $(\dot{+}, (-1))$–duality in the sense of [11, Definition 4.3], the biconjugate of g is the image of g^* under the dual $\mathrm{c}' : (\mathbb{R}^{\triangle})^{X^*} \to (\mathbb{R}^{\triangle})^X$ of c, compare Equation (2.7).

To sum up the preliminary considerations we conclude this section with an extended chain rule for scalar valued functions. The inf–convolution with respect to either inf–addition or sup–addition is denoted as $\dot{\square}$ and $\underset{\cdot}{\square}$, respectively.

Theorem 3.6 (Scalar Chain Rule) *Let Y be another topological linear space with topological dual Y^*, $g : X \to \mathrm{I\!R}^\triangle$, $f : Y \to \mathrm{I\!R}^\triangle$ two functions and $T : X \to Y$, $S : Y \to X$ linear continuous operators.*

(a) For all $x \in X$ define

$$\left(g \Box^{\!\cdot} Sf\right)(x) = \inf_{\bar{x} \in X} \left(g(x - \bar{x}) \dot{+} \inf_{Sy = \bar{x}} f(y) \right).$$

The conjugate of $(g \Box^{\!\cdot} Sf)$ in $x^* \in X^*$ is given by

$$\left(g \Box^{\!\cdot} Sf\right)^*(x^*) = \left(g^* \underset{\cdot}{+} f^* S^*\right)(x^*)$$

(b) For $x^* \in X^*$ define

$$\left(g^* \Box_{\!\cdot} T^* f^*\right)(x^*) = \inf_{\bar{x}^* \in X^*} \left(g^*(x^* - \bar{x}^*) \underset{\cdot}{+} T^* f^*(\bar{x}^*) \right).$$

The conjugate of $(g \dot{+} fT)$ is dominated as follows:

$$\left(g \dot{+} fT\right)^*(x^*) \le \left(g^* \Box_{\!\cdot} T^* f^*\right)(x^*)$$

(c) If additionally g or f is the constant mapping $+\infty$, then for all $x^* \in X^*$ and all $y^* \in Y^*$

$$\begin{aligned} -\infty = \left(g \dot{+} fT\right)^*(x^*) &= \left(g^* \Box_{\!\cdot} T^* f^*\right)(x^*) \\ \left(g \dot{+} fT\right)^*(x^*) &= g^*(x^* - T^* y^*) \underset{\cdot}{+} f^*(y^*). \end{aligned}$$

(d) If $(fT)(x_0) = -\infty$ is satisfied for some $x_0 \in \operatorname{dom} g$ or if both f and g are proper, convex and f is continuous in a point in $T(\operatorname{dom} g)$, then for all $x^* \in X^*$

$$-\infty \ne \left(g \dot{+} fT\right)^*(x^*) = \left(g^* \Box_{\!\cdot} T^* f^*\right)(x^*)$$

and it exists $y^* \in Y^*$ such that

$$\left(g \dot{+} fT\right)^*(x^*) = g^*(x^* - T^* y^*) \dot{+} f^*(y^*).$$

Proof.

(a) By definition $(g \Box^{\!\cdot} Sf)^*(x^*)$ equals

$$\sup_{x \in X} \left(x^*(x) \dot{-} \inf_{\bar{x} \in X} \left(g(x - \bar{x}) \dot{+} \inf_{Sy = \bar{x}} f(y) \right) \right)$$

and by Proposition 3.3 this equals

$$x^*(x) \dot{-} \inf_{\bar{x} \in X} \left(g(x - \bar{x}) \dot{+} \inf_{Sy = \bar{x}} f(y) \right) = x^*(x) \underset{\cdot}{+} \sup_{\bar{x} \in X} \left((-1) g(x - \bar{x}) \underset{\cdot}{+} \sup_{Sy = \bar{x}} (-1) f(y) \right).$$

Thus by Proposition 3.1 we get

$$\left(g \Box^{\!\cdot} Sf\right)^*(x^*) = \sup_{x \in X,\, y \in Y} \left(x^*(x) \underset{\cdot}{+} (-1) g(x - Sy) \underset{\cdot}{+} (-1) f(y) \right)$$

and as $x^*(x) = x^*(x - Sy) + S^* x^*(y)$, again by calculus rules for residuation we get

$$\left(g \Box^{\!\cdot} Sf\right)^*(x^*) = g^*(x^*) \underset{\cdot}{+} (f^* S^*)(x^*).$$

(b) First we consider $T^*y^* = x^*$:

$$(fT)^*(x^*) = \sup_{x\in X}\big(y^*(Tx)\dot{-} f\,(T(x))\big) \le \sup_{y\in Y}\big(y^*(y)\dot{-} f(y)\big) = f^*(y^*).$$

We apply Propositions 3.3 and 3.1 to prove the following for all $x^*, \bar{x}^* \in X^*$.

$$\begin{aligned}\big(g\dot{+} fT\big)^*(x^*) = &\sup_{x\in X}\Big(\big((x^*-\bar{x}^*)(x)\dot{-} g\,(x)\big) \underset{\cdot}{+} \big(\bar{x}^*(x)\dot{-}(fT)\,(x)\big)\Big)\\ \le &g^*(x^*-\bar{x}^*)\underset{\cdot}{+}(fT)^*(\bar{x}^*)\\ \le &g^*(x^*-\bar{x}^*)\underset{\cdot}{+}T^*f^*(\bar{x}^*).\end{aligned}$$

(c) If either function is constant $+\infty$, then its conjugate is constant $-\infty$ and $(g+fT)$, too is constant $+\infty$ as the inf–addition is dominated by $+\infty$. On the other hand, $-\infty$ dominates the sup–addition, thus

$$(g^*\,(x^*-T^*y^*)\underset{\cdot}{+} f^*\,(y^*)) = -\infty$$

is satisfied for all $x^* \in X^*$ and all $y^* \in Y^*$.

(d) This final result is classic for the proper case, compare e.g. [20, Chap. 3, §3.4 Theorem 1]. The improper case is immediate, as in that case both sides of the equation attain the value $+\infty$. □

Sup–addition is dominated by inf–addition and both operators coincide if neither addend is $-\infty$. Therefor, the last statement of Theorem 3.6 can be read as

$$-\infty \ne \big(g\dot{+} fT\big)^*(x^*) = (g^*\underset{\cdot}{\square}\, T^*f^*)\,(x^*)$$

and it exists $y^* \in Y^*$ with $T^*y^* = x^*$ such that

$$\big(g\dot{+} fT\big)^*(x^*) = g^*\,(x^*-T^*y^*)\underset{\cdot}{+} f^*\,(y^*)\,,$$

if the given assumptions are satisfied.

Using the bijective duality $(-1) : \mathbb{R}^\vartriangle \to \mathbb{R}^\triangledown$, the sup–addition in $(\mathbb{R}^\vartriangle)^{X^*}$ can be represented by

$$(g_1^*\underset{\cdot}{+} g_2^*)(x^*) = -1\big((-g_1^*(x^*))\dot{+}(-g_2^*(x^*))\big)\,. \tag{3.11}$$

Theorem 3.6 (a) is a correct version of [32, Theorem 2.3.1 (ix)] or the first part of [20, Chap.3 §3.4. Theorem 1]. Theorem 3.6 (c) and (d) is an extension of the well known chain–rule for proper functions to a chain–rule for extended real–valued functions.

Example 3.7 *Under the assumptions of Theorem 3.6, let $g \equiv +\infty$ and $f(Tx_0) = -\infty$ with $x_0 \in X$. Then for all $x \in X$ and all $x^* \in X^*$*

$$\begin{aligned}\big(g\dot{\square}\, Sf\big)\,(x) = \big(g\dot{+} fT\big)\,(x) = +\infty;\\ g^*(x^*) = -\infty;\quad f^*S^*(x^*) = T^*f^*(x^*) = +\infty.\end{aligned}$$

Therefore, the statements of Theorem 3.6(a) and (c) do not apply with equality for the inf*–addition on the right hand side.*

4 Set–valued functions of epigraphical type

In the sequel, we will consider X, Y and Z to be locally convex separated spaces with topological duals X^*, Y^* and Z^* and investigate on functions from X to a certain subset $\mathcal{P}^\vartriangle$ of the powerset $\mathcal{P}(Z)$. The space $\mathcal{P}^\vartriangle$ will turn out to be a order complete, inf–residuated conlinear space. As was the case for $\mathrm{I\!R}^\vartriangle$ and $\mathrm{I\!R}^\triangledown$, the notion $\vartriangle$ indicates inf–residuation and a dual space $\mathcal{P}^\triangledown$ is denoted with $\triangledown$ to indicate it is sup–residuated. We define an addition in $\mathcal{P}^\vartriangle$ and identify a bijective duality $s : \mathcal{P}^\vartriangle \to \mathcal{P}^\vartriangle$ such that the conjugate under consideration is a $(+, s)$–duality.

On Z, a reflexive and transitive order is given by a convex cone $C \subseteq Z$ with $\{0\} \subsetneq C \subsetneq Z$, setting $z_1 \leq z_2$, iff $z_2 - z_1 \in C$. The negative dual cone of C is C^-, defined by

$$C^- = \{z^* \in Z^* \mid \forall c \in C : \; z^*(c) \leq 0\}$$

and we assume $C^- \setminus \{0\} \neq \emptyset$. To an element $z^* \in Z^*$ we define

$$H(z^*) = \{z \in Z \mid z^*(z) \leq 0\}.$$

Obviously, C is a subset of $H(z^*)$ if $z^* \in C^-$ and $\operatorname{cl} C = \bigcap\limits_{z^* \in C^- \setminus \{0\}} H(z^*)$.

On $\mathcal{P}(Z)$, we introduce an algebraic structure by defining

$$\begin{aligned} \forall A, B \in \mathcal{P}(Z) : \quad A + B &= \{a + b \mid a \in A,\, b \in B\}, \\ \forall t \in \mathrm{I\!R} \setminus \{0\} : \quad tA &= \{ta \mid a \in A\}, \end{aligned}$$

the Minkowsky sum of two sets and the product of a set with a real number $t \neq 0$. By convention, $A + \emptyset = \emptyset + A = \emptyset$ and $t\emptyset = \emptyset$ for all $A \in \mathcal{P}(Z)$ and $t \neq 0$ while $0A = \{0\}$ for all $A \in \mathcal{P}(Z)$. We abbreviate $z + A = \{z\} + A$ and $-A = (-1)A$ as well as $A - B = A + (-1)B$ for $A, B \in \mathcal{P}(Z)$ and $z \in Z$.

The order relation in Z can be extended in two distinct ways by setting

$$\begin{aligned} \forall A, B \in \mathcal{P}(Z) : \quad A \preccurlyeq_C B \quad &\Leftrightarrow B \subseteq A + C; \\ \forall A, B \in \mathcal{P}(Z) : \quad A \curlyeqprec_C B \quad &\Leftrightarrow A \subseteq B - C, \end{aligned}$$

see [13, 16, 23]. Let $A, B \in \mathcal{P}(Z)$, then we attain $A \curlyeqprec_C B$, iff $-B \preccurlyeq_C -A$ or equivalently if $B \preccurlyeq_{-C} A$.

Two sets $A, B \in \mathcal{P}(Z)$ are equivalent with respect to $\preccurlyeq_C$, iff $A \preccurlyeq_C B$ and $B \preccurlyeq_C A$, i.e. if $A + C = B + C$. We identify the set

$$\mathcal{P}^\vartriangle(Z, C) = \{A \in \mathcal{P}(Z) \mid A = A + C\}$$

and abbreviate $\mathcal{P}^\vartriangle = \mathcal{P}^\vartriangle(Z, C)$, if no confusion can arise. In $\mathcal{P}^\vartriangle$, $A \preccurlyeq_C B$ is equivalent to $B \subseteq A$. The set $\mathcal{P}^\vartriangle$ is a complete lattice, infimum and supremum of a nonempty set $\mathcal{M} \subseteq \mathcal{P}^\vartriangle$ are given by

$$\inf_{M \in \mathcal{M}} M = \bigcup_{M \in \mathcal{M}} M \in \mathcal{P}^\vartriangle; \quad \sup_{M \in \mathcal{M}} M = \bigcap_{M \in \mathcal{M}} M \in \mathcal{P}^\vartriangle$$

and by convention $\inf \emptyset = \emptyset$ and $\sup \emptyset = Z$. The greatest element in $\mathcal{P}^{\triangle}$ is $\emptyset$, the smallest Z. For any $\mathcal{M} \subseteq \mathcal{P}^{\triangle}$ and $A \in \mathcal{P}^{\triangle}$, the following properties are satisfied, compare also Proposition 3.1.

$$\begin{aligned} \inf_{M \in \mathcal{M}} (A + M) &= A + \inf_{M \in \mathcal{M}} M; \\ \sup_{M \in \mathcal{M}} (A + M) &\preccurlyeq_C A + \sup_{M \in \mathcal{M}} M. \end{aligned}$$

Altering the multiplication with 0 to $0A = C$ for all $A \in \mathcal{P}^{\triangle}$, the set $\mathcal{P}^{\triangle}$ together with the Minkowsky sum, the altered multiplication with nonnegative reals and the order relation $\supseteq$ is a order complete, inf–residuated conlinear space with neutral elment C. We abbreviate $\mathcal{P}^{\triangle} = (\mathcal{P}^{\triangle}, +, \cdot, \supseteq)$ and

$$\begin{aligned} A \dot{-} B &= \inf \left\{ M \in \mathcal{P}^{\triangle} \mid A \preccurlyeq_C B + M \right\} \\ &= \{ z \in Z \mid A \supseteq B + z \}, \end{aligned}$$

compare [28, Section 4] and the references therein on the usage of the inf–residual of sets. Multiplication with -1 is a duality from $\mathcal{P}^{\triangle}$ to the order complete, sup–residuated conlinear space

$$\mathcal{P}^{\triangledown} = (\{ A - C \mid A \in \mathcal{P}(Z) \}, +, \cdot, \subseteq).$$

With convex duality in mind, the space $\mathcal{P}^{\triangle}$ is more appropriate, a closer study of both extensions can be found in [16, 24].

Definition 4.1 *The graph of a function $g : X \to \mathcal{P}(Z)$ is defined as*

$$\operatorname{graph} g = \{ (x, z) \in X \times Z \mid z \in g(x) \},$$

the domain and epigraph of g are given by

$$\begin{aligned} \operatorname{dom} g &= \{ x \in X \mid g(x) \neq \emptyset \}, \\ \operatorname{epi} g &= \{ (x, z) \in X \times Z \mid z \in g(x) + C \}. \end{aligned}$$

If $g : X \to \mathcal{P}^{\triangle}$, then $\operatorname{graph} g = \operatorname{epi} g$, motivating the notion *epigraphical type function* for functions $g : X \to \mathcal{P}^{\triangle}$.

Definition 4.2 *A function $g : X \to \mathcal{P}^{\triangle}$ is called positively homogeneous, iff* epi g *is a cone, convex or closed, iff* epi g *is convex or closed, subadditive if* epi g *is closed under addition and sublinear, iff* epi g *is a convex cone.*

A function $h : X \to \mathcal{P}^{\triangledown}$ is said to be concave, iff $\operatorname{hypo} h = \operatorname{graph} h$ is convex. Thus, $g : X \to \mathcal{P}^{\triangle}$ is convex, if and only if $-g : X \to \mathcal{P}^{\triangledown}$ is concave.

The set $(\mathcal{P}^{\triangle})^X = \{ g : X \to \mathcal{P}^{\triangle} \}$ is ordered by the point–wise ordering; $g_1 \leq g_2$ is satisfied, iff $g_1(x) \supseteq g_2(x)$ is satisfied for all $x \in X$, or equivalently if $\operatorname{epi} g_1 \supseteq \operatorname{epi} g_2$.

Remark 4.3 *If a function $g : X \to \mathcal{P}^\vartriangle$ is convex or closed, then especially for each $x \in X$ the set $g(x)$ is convex or closed. Defining $(\mathrm{cl}\,\mathrm{co}\,g) : X \to \mathcal{P}^\vartriangle$ by setting $\mathrm{epi}\,(\mathrm{cl}\,\mathrm{co}\,g) = \mathrm{cl}\,\mathrm{co}\,(\mathrm{epi}\,g)$, the function $(\mathrm{cl}\,\mathrm{co}\,g)$ maps into the set $\mathcal{Q}^\vartriangle$ of all convex, closed sets $A \in \mathcal{P}(Z)$ with $A = \mathrm{cl}\,\mathrm{co}\,(A+C)$. Altering the addition of sets to $A \oplus B = \mathrm{cl}\,(A+B)$ and multiplication with 0 to $0A = \mathrm{cl}\,C$, then $\mathcal{Q}^\vartriangle$ with the altered addition, multiplication and the order relation $\supseteq$ is a $\inf$–residuated order complete conlinear space. This space has been used as image space in [14, 29].*

Definition 4.4 *A function $g : X \to \mathcal{P}^\vartriangle$ is called proper, iff $\mathrm{dom}\,g \neq \emptyset$ and there is no $x \in X$ with $g(x) = Z$. A function g is called z^*–proper with $z^* \in Z^*$, iff $\mathrm{dom}\,g \neq \emptyset$ and for all $x \in \mathrm{dom}\,g$ it holds $(g(x) - H(z^*)) \setminus (g(x) + H(z^*)) \neq \emptyset$.*

Obviously, no function is 0-proper. If a function $g : X \to \mathcal{P}^\vartriangle$ is z^*–proper for some $z^* \in Z^*$, then g is proper and $z^* \in C^- \setminus \{0\}$. For a closed convex function, even more can be said.

Proposition 4.5 *Let $g : X \to \mathcal{Q}^\vartriangle$ be a closed and convex function, then g is proper if and only if there exists $z^* \in C^- \setminus \{0\}$ such that g is z^*–proper.*

Proof. A closed convex function $g : X \to \mathcal{Q}^\vartriangle$ is proper, iff there is $x_0 \in \mathrm{dom}\,g$ and $z_0 \in Z$ such that $(x_0, z_0) \notin \mathrm{epi}\,g$. By a separation argument, there is $(x^*, z^*) \in X^* \times C^- \setminus \{0\}$ such that

$$(x^*, -z^*)(x_0, z_0) \leq \inf_{(x,z)\in \mathrm{epi}\,g} (x^*, -z^*)(x, z)$$

and thus $-x^*(x) + (x^*, -z^*)(x_0, z_0) \leq \inf\limits_{z\in g(x)} -z^*(z)$ for all $x \in X$, which is equivalent to g being z^*–proper. □

As in the scalar case, if a closed convex function $g : X \to \mathcal{P}^\vartriangle$ is improper, then $g(x) = Z$ holds for all $x \in \mathrm{dom}\,g$ and $\mathrm{dom}\,g$ is a closed convex set in X. Likewise, if a closed convex function is z^*–improper, then $(g(x) - H(z^*)) \setminus (g(x) + H(z^*)) = \emptyset$ is satisfied for all $x \in \mathrm{dom}\,g$ and $\mathrm{dom}\,g$ is closed and convex, compare [14, Proposition 5].

The set of all functions $g : X \to \mathcal{P}^\vartriangle$ equipped with the point–wise addition, multiplication, order relation and inf–residuation is a inf–residuated order complete conlinear space. We denote the inf–convolution of $f, g : X \to \mathcal{P}^\vartriangle$ by

$$\forall x \in X : \quad (f \Box g)(x) = \inf_{y\in X} \left(f(x-y) + g(y)\right) \tag{4.1}$$

and for another locally convex separated space Y, $f : Y \to \mathcal{P}^\vartriangle$ and $g : X \to \mathcal{P}^\vartriangle$ we denote

$$(fT)(x) = f(Tx); \quad (Tg)(y) = \inf_{Tx=y} g(x) \tag{4.2}$$

for all $x \in X$ and a linear continuous operator $T : X \to Y$. The definitions in (4.1) and (4.2) can be found in [14, 29].

5 Scalarization of set–valued functions

Definition 5.1 *Let $g : X \to \mathcal{P}^\triangle$ and $\phi : Z \to \overline{\mathrm{IR}}$ be two functions. The scalarization of g with respect to ϕ is defined by*

$$\forall x \in X : \quad \varphi_{g,\phi}(x) = \inf_{z \in g(x)} -\phi(z).$$

If $\operatorname{dom}\phi = Z$, then $\operatorname{dom}\varphi_{g,\phi} = \operatorname{dom} g$. If $\phi \in Z^*$, then $\varphi_{g,\phi}(x)$ equals the negative support function of ϕ at $g(x)$,

$$\varphi_{g,z^*}(x) = -\sup_{z \in g(x)} z^*(z).$$

Especially, in the case $\phi \equiv 0$ we obtain

$$\forall x \in X : \quad \varphi_{g,0}(x) = I_{\operatorname{dom} g}(x), \tag{5.1}$$

thus $\varphi_{g,0}(x) = 0$ for all $x \in \operatorname{dom} g$, compare also Equation (3.6).

If $g(x) = \operatorname{cl}\operatorname{co}(g(x)) \in \mathcal{Q}^\triangle$, then by a separation argument for all $x \in X$

$$g(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \{z \in Z \mid \varphi_{g,z^*}(x) \le -z^*(z)\}. \tag{5.2}$$

The scalarization $\varphi_{g,0}$ can be omitted, as $\{z \in Z \mid \varphi_{g,0}(x) \le 0\} = Z$ holds for all $x \in \operatorname{dom} g$ and $\{z \in Z \mid \varphi_{g,z^*}(x) \le 0\} = \emptyset$ for all $z^* \in C^-$ and $x \notin \operatorname{dom} g$.

A function $g : X \to \mathcal{Q}^\triangle$ is z^*-proper for $z^* \in C^- \setminus \{0\}$, iff $\varphi_{g,z^*} : X \to \overline{\mathrm{IR}}$ is proper. Thus, g is proper, if there is $z^* \in C^- \setminus \{0\}$ such that φ_{g,z^*} is proper. Also, g is convex, positively homogeneous or subadditive, if for all $z^* \in C^-$ the scalarization φ_{g,z^*} has the corresponding property. Closedness is not as immediate, as the following example shows. However, if all scalarizations φ_{g,z^*} with $z^* \in C^- \setminus \{0\}$ are closed, then g is closed.

Example 5.2 *Let the set IR^2 be ordered by the usual ordering cone $C = \mathrm{IR}^2_+$, $z^* = (0, -1)$ and $g : \mathrm{IR} \to \mathrm{IR}^2$ be defined as $g(x) = \left\{(\frac{1}{x}, 0)\right\} + C$, if $x > 0$ and $g(x) = \emptyset$, else. Thus, $\varphi_{g,z^*}(0) = +\infty$, while $\varphi_{g,z^*}(x) = 0$ holds for all $x > 0$ and therefore $\operatorname{cl}\varphi_{g,z^*}(0) = 0$.*

Proposition 5.3 *Let $g : X \to \mathcal{P}^\triangle$ be a function, then for all $x \in X$ the following is satisfied.*

$$(\operatorname{cl}\operatorname{co} g)(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \{z \in Z \mid \operatorname{cl}\operatorname{co}\varphi_{g,z^*}(x) \le -z^*(z)\}.$$

Proof. For simplicity suppose that g is a closed convex function. The images $g(x)$ are elements of $\mathcal{Q}^\triangle$ and because of (5.2) it is left to prove

$$g(x) \supseteq \bigcap_{z^* \in C^- \setminus \{0\}} \{z \in Z \mid \operatorname{cl}\operatorname{co}\varphi_{g,z^*}(x) \le -z^*(z)\} \tag{5.3}$$

for all $x \in X$. If g is improper, then $\operatorname{dom} g$ is closed and convex, thus $\operatorname{dom} g = \operatorname{dom}\operatorname{cl}\operatorname{co}\varphi_{g,z^*}$ holds for all $z^* \in C^-$. If $z^* \in C^- \setminus \{0\}$, then $\varphi_{g,z^*}(x) = -\infty$ holds for $x \in \operatorname{dom} g$ and $\varphi_{g,z^*}(x) = +\infty$, else. In this case, (5.3) is immediate.

Suppose g is proper, $(x_0, z_0) \notin \operatorname{epi} g$. By a separation argument it exists $(x^*, z^*) \in X^* \times Z^*$ and $t \in \mathrm{I\!R}$ such that

$$-x^*(x_0) - z^*(z_0) < t < -x^*(x) - z^*(z)$$

for all $(x, z) \in \operatorname{epi} g$. Thus $z^* \in C^-$ and an affine minorant of φ_{g,z^*} is defined through x^* and t, separating $(x_0, -z^*(z_0))$ from the epigraph of φ_{g,z^*}, proving

$$g(x) = \bigcap_{z^* \in C^-} \{z \in Z \mid \operatorname{cl}\operatorname{co}\varphi_{g,z^*}(x) \le -z^*(z)\}$$

for all $x \in X$.

Next, chose $(\bar{x}, \bar{z}) \in X \times Z$ such that $\bar{x} \in \operatorname{dom} g$ and $\bar{z} \notin g(\bar{x})$. As $\bar{x} \in \operatorname{dom} g$, it exists $(\bar{x}^*, \bar{z}^*, \bar{t}) \in X^* \times (C^- \setminus \{0\}) \times \mathrm{I\!R}$ with

$$-\bar{x}^*(\bar{x}) - \bar{z}^*(\bar{z}) < \bar{t} < -\bar{x}^*(x) - \bar{z}^*(z)$$

for all $(x, z) \in \operatorname{epi} g$. If $(\bar{x}^*, \bar{z}^*)$ separates (x_0, z_0) from $\operatorname{epi} g$, then there is nothing more to prove. Otherwise, we can chose $s > 0$ such that

$$-(x^* + s\bar{x}^*)(x_0) - (z^* + s\bar{z}^*)(z_0) < -(x^* + s\bar{x}^*)(x) - (z^* + s\bar{z}^*)(z)$$

for all $(x, z) \in \operatorname{epi} g$. By assumption $(z^* + s\bar{z}^*) \in C^- \setminus \{0\}$ is fulfilled, thus (5.3) is proven. □

As an immediate corollary we get

Corollary 5.4 *Let $f, g : X \to \mathcal{P}^\triangle$ be two functions. Then $(\operatorname{cl}\operatorname{co} f) \le (\operatorname{cl}\operatorname{co} g)$ is satisfied, iff for all $z^* \in C^- \setminus \{0\}$ the function $(\operatorname{cl}\operatorname{co}\varphi_{f,z^*})$ is a minorant of $(\operatorname{cl}\operatorname{co}\varphi_{g,z^*})$.*

Moreover, $\operatorname{cl}\operatorname{co} g : X \to \mathcal{P}^\triangle$ is either proper or constant Z if and only if the following equality is satisfied for all $x \in X$.

$$(\operatorname{cl}\operatorname{co} g)(x) = \bigcap_{\substack{\operatorname{cl}\operatorname{co}\varphi_{g,z^*} \\ \text{is proper,} \\ z^* \in C^- \setminus \{0\}}} \{z \in Z \mid \operatorname{cl}\operatorname{co}\varphi_{g,z^*}(x) \le -z^*(z)\}.$$

Proposition 5.5 *Let I be an index set, $f, g, g_i : X \to \mathcal{P}^\triangle$ be functions for all $i \in I$, Y another locally convex separated space with dual Y^*, $T : X \to Y$ a linear continuous operator and $h : Y \to \mathcal{P}^\triangle$ a function. Let $z^* \in C^-$, then the following formulas are true.*

(a) $\forall x \in X : \quad \varphi_{f+g,z^*}(x) = \varphi_{f,z^*}(x) \dot{+} \varphi_{g,z^*}(x).$

(b) $\forall x \in X : \quad \varphi_{hT,z^*}(x) = \varphi_{h,z^*}T(x).$

(c) $\forall x \in X : \quad \varphi_{\inf\limits_{i\in I} g_i,z^*}(x) = \inf\limits_{i\in I} \varphi_{g_i,z^*}(x).$

Proof. (a) and (b) are immediate from the Definition 5.1 while (c) holds true by $(\inf\limits_{i\in I} g_i)(x) = \bigcup\limits_{i\in I} g_i(x)$. □

Combining (a) and (c) from Proposition 5.5 one can derive

$$\forall x \in X : \quad \varphi_{f\Box g,z^*}(x) = (\varphi_{f,z^*}\Box\varphi_{g,z^*})(x) \tag{5.4}$$

and by (b) and (c)

$$\forall x \in X : \quad \varphi_{Th,z^*}(x) = T\varphi_{h,z^*}(x). \tag{5.5}$$

Proposition 5.6 *Let $f, g : X \to \mathcal{P}^\triangle$ and $g_i; X \to \mathcal{P}^\triangle$ be functions for all $i \in I$, then for all $z^* \in C^-$ and all $x \in X$*

$$\sup_{i\in I} \varphi_{g_i,z^*}(x) \leq \varphi_{\sup\limits_{i\in I} g_i,z^*}(x);$$
$$\varphi_{f,z^*}(x) \mathbin{\dot-} \varphi_{g,z^*}(x) \leq \varphi_{(f \mathbin{\dot-} g),z^*}(x).$$

If $(\sup\limits_{i\in I} g_i)(x) = \bigcap\limits_{i\in I} \{z \in Z \mid \varphi_{g_i,z^*}(x) \leq -z^*(z)\}$ and $f(x) = \{z \in Z \mid \varphi_{f,z^*}(x) \leq -z^*(z)\}$ respectively, then the both inequalities are satisfied with equality.

Proof. Let $z \in (f \mathbin{\dot-} g)(x)$, or equivalently $g(x)+\{z\} \subseteq f(x)$. In this case $(\varphi_{f,z^*} \mathbin{\dot-} \varphi_{g,z^*})(x) \leq -z^*(z)$ is fulfilled. If $z \in \sup\limits_{i\in I} g_i(x)$, then $\sup\limits_{i\in I} \varphi_{g_i,z^*}(x) \leq -z^*(z)$, thus both inequalities are proven.

If $(\sup\limits_{i\in I} g_i)(x) = \bigcap\limits_{i\in I} \{z \in Z \mid \varphi_{g_i,z^*}(x) \leq -z^*(z)\}$ is assumed, then

$$\inf\left\{-z^*(z) \mid z \in (\sup_{i\in I} g_i)(x)\right\} = \sup_{i\in I} \varphi_{g_i,z^*}(x).$$

If $f(x) = \{z \in Z \mid \varphi_{f,z^*}(x) \leq -z^*(z)\}$ is assumed for all $x \in X$, then $z \in (f \mathbin{\dot-} g)(x)$ is satisfied iff $\varphi_{f,z^*}(x) \leq -z^*(z) + \varphi_{g,z^*}(x)$, thus in this case

$$\varphi_{(f \mathbin{\dot-} g),z^*}(x) \leq \varphi_{f,z^*}(x) \mathbin{\dot-} \varphi_{g,z^*}(x).$$

□

If $\mathrm{c} : (\mathcal{P}^\triangle)^X \to (\mathcal{P}^\triangle)^Y$ is a duality and for all $g \in (\mathcal{P}^\triangle)^X$ it holds

$$\mathrm{c}\,(g)(x) = \left\{z \in Z \mid \varphi_{\mathrm{c}\,(g),z^*}(x) \leq -z^*(z)\right\},$$

then by Proposition 5.5 (c) and Proposition 5.6 the mapping

$$\forall g : X \to \mathcal{P}^\triangle, \forall z^* \in C^- : \quad \varphi_{g,z^*} \mapsto \varphi_{\mathrm{c}\,(g),z^*}$$

is a duality from $(\bar{\mathbb{R}})^X$ to $(\bar{\mathbb{R}})^X$

Definition 5.7 *Let $f : X \to \overline{\mathbb{R}}$ and $\phi : Z \to \overline{\mathbb{R}}$ be two functions. The set–valued function $S_{(f,\phi)} : X \to \mathcal{P}^\triangle$ is defined by*

$$\forall x \in X : \quad S_{(f,\phi)}(x) = \{z \in Z \mid f(x) \leq -\phi(z)\}.$$

Obviously, if f_1 is a minorant of $f_2 : X \to \overline{\mathrm{IR}}$, then

$$S_{(f_1,\phi)}(x) \supseteq S_{(f_2,\phi)}(x)$$

is met for all $x \in X$ and all $\phi : Z \to \overline{\mathrm{IR}}$.

Each function $f : X \to \overline{\mathrm{IR}}$ is dominated by $\varphi_{S_{(f,\phi)},\phi}$. If $\phi(X) \supseteq \mathrm{IR}$, then equality holds true.

Proposition 5.8 *Let $f : X \to \overline{\mathrm{IR}}$, $z^* \in Z^*$ and $g : X \to \mathcal{P}^\triangle$. Then $S_{(f,z^*)}$ is a minorant of g if and only if f is a minorant of φ_{g,z^*}.*

PROOF. First, let $S_{(f,z^*)}(x) \supseteq g(x)$ be assumed for all $x \in X$. Then

$$\forall x \in X : \quad f(x) \le \varphi_{S_{(f,z^*)},z^*}(x) \le \varphi_{g,z^*}(x)$$

is satisfied, f is a minorant of φ_{g,z^*}. On the other hand, if f is dominated by φ_{g,z^*}, then

$$\forall x \in X : \quad S_{(f,z^*)}(x) \supseteq S_{(\varphi_{g,z^*}),z^*}(x) \supseteq g(x).$$

□

Proposition 5.9 *Let $g : X \to \mathcal{P}^\triangle$ be a function, $x \in X$ and $z^* \in C^-$. It holds*

$$S_{(\varphi_{g,z^*},z^*)}(x) = \mathrm{cl}\,(g(x) + H(z^*))\,;$$
$$\mathrm{cl}\,\mathrm{co}\,(g(x)) = \bigcap_{z^* \in C^- \setminus \{0\}} S_{(\varphi_{g,z^*},z^*)}(x).$$

PROOF. By definition,

$$S_{(\varphi_{g,z^*},z^*)}(x) = \left\{ z \in Z \mid \inf_{\bar{z} \in g(x)} (-z^*(\bar{z})) \le -z^*(z) \right\}$$

If $z^* = 0$, then $S_{(\varphi_{g,z^*},z^*)}(x) = Z$ for $x \in \mathrm{dom}\, g$ and $S_{(\varphi_{g,z^*},z^*)}(x) = \emptyset$, else. If $z^* \ne 0$, then the set $S_{(\varphi_{g,z^*},z^*)}(x)$ is a closed affine half–space in Z with

$$(g(x) + \{z \in Z \mid 0 \le -z^*(z)\}) \subseteq S_{(\varphi_{g,z^*},z^*)}(x)$$

and by a separation argument both equalities are proven. □

Corollary 5.10 *Let $f : X \to \overline{\mathrm{IR}}$ be a function $x \in X$. Then*

$$\begin{array}{lclcl} f(x) \le 0 & \Leftrightarrow & S_{(f,0)}(x) = Z & \Leftrightarrow & \varphi_{S_{(f,0)},0}(x) = 0; \\ 0 < f(x) & \Leftrightarrow & S_{(f,0)}(x) = \emptyset & \Leftrightarrow & \varphi_{S_{(f,0)},0}(x) = +\infty. \end{array}$$

The scalarization $\varphi_{S_{(f,0)},0}$ is the indicator function of the sublevelset of f at 0. Especially for $(x^*, r) \in X^* \times \mathrm{IR}$ and $z^* \in C^-$ it holds

$$\forall x \in X : \quad S_{(\hat{x}^*_r,z^*)}(x) = S_{(x^*_r,0)}(x) = \begin{cases} Z, & \text{if } x^*_r(x) \le 0; \\ \emptyset, & \text{else.} \end{cases}$$

The scalarization $\varphi_{S_{(x^*_r,0)},0}$ is the indicator function of $\mathrm{dom}\,\hat{x}^*_r$, compare Formula (5.1), while $\hat{x}^*_r = \varphi_{S_{(\hat{x}^*_r,z^*)},z^*}$ is satisfied if $z^* \ne 0$, compare Formulas (5.1) and (3.6).

For $x^* \in X^*$, $z^* \in C^-$, $r \in \mathrm{I\!R}$ and $x \in X$ we denote

$$S_{(x^*,r,z^*)}(x) = S_{((x^*_r),z^*)}(x).$$

and abbreviate $S_{((x^*,z^*)}(x) = S_{(x^*,0,z^*)}(x)$.

A function of the type $S_{(x^*,r,z^*)} : X \to \mathcal{P}^\triangle$ with $(x^*, r, z^*) \in X^* \times \mathrm{I\!R} \times C^-$ is called conaffine. If additionally $r = 0$, then $S_{(x^*,z^*)} : X \to \mathcal{P}^\triangle$ is called conlinear. If $z^* \neq 0$, then $S_{(x^*,r,z^*)}$ is proper. If $z^* = 0$, then $S_{(x^*,r,0)}(x) = Z$ for all $x \in \operatorname{dom} \hat{x}^*_r$ and $S_{(x^*,r,0)}(x) = \emptyset$ for $x \notin \operatorname{dom} \hat{x}^*_r$.

Proposition 5.11 *Let $x, y \in X$, $z^* \in C^-$, $x^*, y^* \in X^*$, $r \in \mathrm{I\!R}$ and $t > 0$ then the following statements are true.*

(a) The function $S_{(x^*,z^*)} : X \to \mathcal{Q}^\triangle$ is sublinear and closed and

$$S_{(x^*,z^*)}(x) = S_{(-x^*,z^*)}(-x) = H(z^*) \mathbin{-\!\!\!^{\cdot}} S_{(x^*,z^*)}(-x).$$

(b)

$$S_{(x^*+y^*,r,z^*)}(x) = \operatorname{cl} \bigcup_{s \in \mathrm{I\!R}} \left(S_{(x^*,s,z^*)}(x) + S_{(y^*,r-s,z^*)}(x) \right)$$
$$S_{(tx^*,tr,z^*)}(x) = S_{(x^*,\frac{1}{t}z^*)}(x) = S_{(x^*,tr,z^*)}(tx).$$

(c) If additionally $z^* \neq 0$ and $z_0 \in Z$ with $z^*(z_0) = 1$, then

$$S_{(x^*,r,z^*)}(x) = S_{(x^*,z^*)}(x) + \{rz_0\}.$$

Moreover, the following additivity properties are met.

$$S_{(x^*+y^*,r,z^*)}(x) = S_{(x^*,z^*)}(x) + S_{(y^*,z^*)}(x) + \{rz_0\}$$
$$S_{(x^*,r,z^*)}(x+y) = S_{(x^*,z^*)}(x) + S_{(x^*,z^*)}(y) + \{rz_0\}.$$

The function $S_{(x^*,z^*)}$ is proper for all $x^* \in X^*$ and $\operatorname{dom} S_{(x^*,z^*)} = X$.

Proof.

(a) By definition, $S_{(x^*+y^*,r,z^*)}(x)$ is equal to the set $\{z \in Z \mid x^*(x) + y^*(x) - r \leq -z^*(z)\}$, thus $t \cdot S_{(x^*,z^*)}(x) = S_{(x^*,z^*)}(tx)$ is fulfilled for all $t > 0$, the epigraph of $S_{(x^*,z^*)}$ is closed and

$$S_{(x^*,z^*)}(x+y) \supseteq S_{(x^*,z^*)}(x) + S_{(x^*,z^*)}(y);$$
$$S_{(x^*,z^*)}(x) = S_{(-x^*,z^*)}(-x).$$

Moreover,

$$H(z^*) \mathbin{-\!\!\!^{\cdot}} S_{(x^*,z^*)}(-x) = \{z \in Z \mid 0 \leq x^*(-x) - z^*(z)\} = S_{(x^*,z^*)}(x).$$

(b) By definition, $z \in S_{(x^*+y^*,r,z^*)}(x)$ is true if and only if it exists $s \in \mathrm{I\!R}$ such that there are $z_1, z_2 \in Z$ with $z_1 + z_2 = z$ and

$$x^*(x) - s \leq -z^*(z_1),\; y^*(x) - r + s \leq -z^*(z_2).$$

The multiplication formula is proven by direct

(c) If $z^* \neq 0^*$ and $z_0 \in Z$ with $z^*(z_0) = 1$, then

$$S_{(x^*,z^*)}(x) + \{rz_0\} = \{z + rz_0 \in Z \mid x^*(x) \leq -z^*(z)\} = S_{(x^*,r,z^*)}(x),$$

thus

$$S_{(x^*,s,z^*)}(x) + S_{(y^*,r-s,z^*)}(x) = S_{(x^*,z^*)}(x) + S_{(y^*,z^*)}(x) + \{rz_0\}\,.$$

By definition,

$$S_{(x^*,z^*)}(x) = \{z \in Z \mid x^*(x) \leq -z^*(z)\}$$

and therefor $S_{(x^*,z^*)}(x) \neq \emptyset$ is satisfied for all $x \in X$ and $S_{(x^*,z^*)}$ is proper and additive. □

In parts, Proposition 5.11 can be found in [14, Proposition 6, 7]. If $x^* \in X^*$ and $z^* \in C^-$, then $S_{(x^*,z^*)}(0) = H(z^*)$, while $0 \cdot S_{(x^*,z^*)}(x) = C$, if $S_{(x^*,z^*)}$ is interpreted as a mapping from X to $\mathcal{P}^{\triangle}$. The set $H(Z^*)$ is the neutral element in the space

$$\mathcal{P}^{\triangle}(Z, H(z^*)) = \{A \in \mathcal{P}(Z) \mid A = A + H(z^*)\}$$

and $S_{(x^*,z^*)}(x) \in \mathcal{P}^{\triangle}(Z, H(z^*))$ is met for all $x \in X$. Interpreting the function $S_{(x^*,z^*)}$ to map into the space $P^{\triangle}(Z, H(z^*))$, it holds $0 \cdot S_{(x^*,z^*)}(x) = H(z^*)$, the neutral element in $P^{\triangle}(Z, H(z^*))$ and the set

$$\left\{S_{(x^*,z^*)} : X \to P^{\triangle}(Z, H(z^*)) \mid x^* \in X^*\right\}$$

is isomorph to the set X^* if $z^* \neq 0$. The set

$$\left\{S_{(x^*,0)} : X \to P^{\triangle}(Z, H(0^*)) \mid x^* \in X^*\right\}$$

is one-to-one to the set $\hat{X}^*$.

Remark 5.12 *Let $g : X \to \mathcal{P}^{\triangle}$ be a function, $x^* \in X^*$, $r \in \mathrm{I\!R}$ and $z^* \in C^-$. The function $S_{(x^*,r,z^*)}$ is a conaffine minorant of g if and only if x^*_r is a minorant of φ_{g,z^*}, compare Proposition 5.8. Especially,*

$$S_{(\hat{x}^*,r,z^*)}(x) = S_{(x*,r,0)}(x) \supseteq g(x) \quad \Leftrightarrow \quad x^*(x) \dot{-} r \leq I_{\mathrm{dom}\, g}(x),$$

compare Corollary 5.10 and Equations (3.6) and (5.1).

Proposition 5.13 *Let $g : X \to \mathcal{P}^\vartriangle$, $z^* \in C^- \setminus \{0\}$, $x^* \in X^*$ and $r \in \mathrm{I\!R}$. Then x_r^* is a minorant of φ_{g,z^*} if and only if $S_{(x^*,r,z^*)}$ is a minorant of g. The closed convex hull of g is proper or constant $\emptyset$ or Z, if and only if*

$$\forall x \in X: \quad (\operatorname{cl}\operatorname{co} g)(x) = \bigcap_{\substack{x_r^* \leq \varphi_{g,z^*},\\ z^* \in C^- \setminus \{0\}}} S_{(x^*,r,z^*)}(x).$$

If $(\operatorname{cl}\operatorname{co} g) : X \to \mathcal{P}^\vartriangle$ is improper, then

$$\forall x \in X: \quad (\operatorname{cl}\operatorname{co} g)(x) = \bigcap_{x_r^* \leq \varphi_{g,0}} S_{(x^*,r,0)}(x).$$

Proof. Without loss of generality, assume g to be closed and convex. By Corollary 5.4 and Equation (3.8)

$$\begin{aligned} g(x) &= \bigcap_{z^* \in C^- \setminus \{0\}} \{z \in Z \mid \operatorname{cl}\operatorname{co} \varphi_{g,z^*}(x) \leq -z^*(z)\} \\ (\operatorname{cl}\operatorname{co} \varphi_{g,z^*})(x) &= \sup_{\substack{\xi_r \leq \varphi_{g,z^*};\\ (\xi,r) \in (X^\vartriangle \times \mathrm{I\!R})}} \xi_r(x), \end{aligned}$$

while Remark 5.12 states

$$S_{(\xi_r,z^*)}(x) \supseteq g(x) \quad \Leftrightarrow \quad \xi_r(x) \leq \varphi_{g,z^*}(x)$$

and by Equations (3.6) and (5.1), $\hat{x}_r^*(x) \leq \varphi_{g,z^*}(x)$ is equivalent to

$$x^*(x) - r \leq I_{\operatorname{dom} g}(x) = \varphi_{g,z^*}(x).$$

Thus,

$$\begin{aligned} g(x) &= \bigcap_{z^* \in C^- \setminus \{0\}} \{z \in Z \mid \operatorname{cl}\operatorname{co} \varphi_{g,z^*}(x) \leq -z^*(z)\} \\ &= \bigcap_{\substack{\xi_r \leq \varphi_{g,z^*};\\ (\xi,r) \in (X^\vartriangle \times \mathrm{I\!R});\\ z^* \in C^- \setminus \{0\}}} \{z \in Z \mid \xi_r(x) \leq -z^*(z)\} \\ &= \bigcap_{\substack{x_r^* \leq \varphi_{g,z^*};\\ (x^*,r) \in (X^* \times \mathrm{I\!R});\\ z^* \in C^-}} S_{(x^*,r,z^*)}(x). \end{aligned}$$

Applying Corollary 5.4, the first statement is immediate, while the second is true, as the only proper scalarization of an improper function g is $\varphi_{g,0}$. □

Notice that the scalarizations of $g : X \to \mathcal{P}^\vartriangle$ used Proposition 5.13 are either proper or constant $+\infty$ or $-\infty$ and thus also the affine minorants needed in the representation are proper. Alternatively, the representation can be done excluding the 0–scalarization $\varphi_{g,0}$, compare [18] but at the cost of properness of the scalarizations and thus also of the affine minorants. The same is true for the following theorem, which is a corollary of Remark 5.12 and Proposition 5.13.

Theorem 5.14 *Let $g : X \to \mathcal{P}^\triangle$ be a convex and closed function. Then g is the point–wise supremum of its conaffine minorants. The function g is proper or constant $\emptyset$ or Z, if and only if it is the point–wise supremum of its proper conaffine minorants,*

$$\forall x \in X : \quad g(x) = \bigcap_{\substack{S_{(x^*,r,z^*)} \preccurlyeq_C g, \\ z^* \in C^- \setminus \{0\}}} S_{(x^*,r,z^*)}(x). \tag{5.6}$$

Otherwise, g is the point–wise supremum of its improper conaffine minorants,

$$\forall x \in X : \quad g(x) = \bigcap_{S_{(x^*,r,0)} \preccurlyeq_C g} S_{(x^*,r,0)}(x). \tag{5.7}$$

The statement of Theorem 5.14 has been proven for the proper case in [14, Theorem 1], while in [18, Theorem 5.30] a representation formula with improper conaffine minorants is proven.

Example 5.15 *[14, Proposition 8] Let $\bar{g} : X \to Z$ be a single–valued function with the epigraphical extension $g(x) = \{\bar{g}(x)\} \oplus C$ for all $x \in X$ and $T : X \to Z$ is a linear continuous operator. Then for all $x \in X$*

$$\varphi_{g,z^*}(x) = -z^*(\bar{g}(x)); \quad S_{(-T^*z^*,z^*)}(x) = \{Tx\} + H(z^*)$$

is satisfied for all $z^* \in C^- \setminus \{0\}$. Moreover, $T(x) + z_0 \leq \bar{g}(x)$ is met if and only if for all $x \in X$ and all $z^* \in C^- \setminus \{0\}$

$$S_{(-T^*z^*,z^*)}(x) + \{z_0\} \supseteq g(x).$$

6 Conjugation of set–valued functions

Definition 6.1 *The conjugate of a function $g : X \to \mathcal{P}^\triangle$ is $g^* : X^* \times C^- \to \mathcal{P}^\triangle$, defined by*

$$g^*(x^*, z^*) = \bigcap_{x \in X} \left(S_{(x^*,z^*)}(x) \dot{-} g(x) \right)$$

for all $(x^*, z^*) \in X^* \times C^-$.

Proposition 6.2 *If $g : X \to \mathcal{P}^\triangle$ is a function and $x^* \in X^*$, $z^* \in C^-$, then*

$$g^*(x^*, z^*) = \{z \in Z \mid (\varphi_{g,z^*})^*(x^*) \leq -z^*(z)\}$$

is satisfied. If additionally $z^* \neq 0$, then

$$(\varphi_{g,z^*})^*(x^*) = \varphi_{g^*(\cdot,z^*),z^*}(x^*)$$

while

$$\left(\varphi_{g,0}\right)^* (x^*) = \sup_{x \in \operatorname{dom} g} x^*(x) \tag{6.1}$$

$$\varphi_{g^*(\cdot,0),0}(x^*) = \begin{cases} 0, & \text{if } \left(\varphi_{g,0}\right)^* (x^*) \leq 0; \\ +\infty, & \text{else.} \end{cases} \tag{6.2}$$

Proof. Let $z^* \in C^- \setminus \{0\}$. Then $\varphi_{S_{(x^*,z^*)},z^*}(x) = x^*(x)$ and by Proposition 5.6 we get

$$\varphi_{g^*(\cdot,z^*),z^*}(x^*) = \sup_{x\in X}\left(x^*(x) \dot{-} \varphi_{g,z^*}(x)\right) = (\varphi_{g,z^*})^*(x^*);$$
$$S_{(x^*,z^*)}(x) \dot{-} g(x) = \left\{z \in Z \mid \left(x^*(x) \dot{-} \varphi_{g,z^*}(x)\right) \leq -z^*(z)\right\}$$

for all $x \in X$ and all $x^* \in X^*$ and

$$g^*(x^*, z^*) = \{z \in Z \mid (\varphi_{g,z^*})^*(x^*) \leq -z^*(z)\} \tag{6.3}$$

holds true. Finally, $\varphi_{g,0}(x) = I_{\operatorname{dom} g}(x)$ proves

$$(\varphi_{g,0})^*(x^*) = \sup_{x\in\operatorname{dom} g} x^*(x)$$

and

$$\begin{aligned} g^*(x^*, 0) &= \bigcap_{x\in X}\{z \in Z \mid g(x) + z \subseteq \{z \in Z \mid x^*(x) \leq 0\}\} \\ &= \begin{cases} Z, & \text{if } x^* \leq I_{\operatorname{dom} g}; \\ \emptyset, & \text{else.} \end{cases} \end{aligned}$$

□

The situation in Equation (6.1) is the same as described in Corollary 5.10. Alternatively, the following representation can be proven for all $x^* \in X^*$:

$$\varphi_{g^*(\cdot,0),0}(x^*) = I_{\left\{x^*\in X^* \mid x^*\leq I_{\operatorname{dom} g}\right\}}(x^*), \tag{6.4}$$

compare also Equation (5.1).

Proposition 6.3 *Let $g : X \to \mathcal{P}^\triangle$ be a function, then the conjugate of g and the conjugate of the closed convex hull of g coincide,*

$$\forall x^* \in X^*, \forall z^* \in C^- : \quad (\operatorname{cl}\operatorname{co} g)^*(x^*, z^*) = g^*(x^*, z^*).$$

Proof. By Proposition 5.3,

$$(\operatorname{cl}\operatorname{co}\varphi_{g,z^*})(x) \leq (\operatorname{cl}\operatorname{co}\varphi_{\operatorname{cl}\operatorname{co} g,z^*})(x),$$

thus for all $x \in X$ and $z^* \in C^-$

$$\operatorname{cl}\operatorname{co}\varphi_{g,z^*}(x) = \operatorname{cl}\operatorname{co}\varphi_{\operatorname{cl}\operatorname{co} g,z^*}(x). \tag{6.5}$$

By Equation (6.3)

$$(\operatorname{cl}\operatorname{co} g)^*(x^*, z^*) = \{z \in Z \mid (\varphi_{\operatorname{cl}\operatorname{co} g,z^*})^*(x^*) \leq -z^*(z)\}$$

is satisfied for all $x^* \in X^*$ and all $z^* \in C^-$. By Equation (6.5) we can conclude $(\operatorname{cl}\operatorname{co} g)^*(x^*, z^*) = g^*(x^*, z^*)$. □

In contrast to the present approach, the (negative) conjugate in [14] is defined as a $Q^{\triangle}$–valued function via an infimum rather than a supremum and thus avoiding a difference operation on the power set $\mathcal{P}(Z)$. In [29, 18, 30], the same idea as in Definition 6.1 has been used. In [29], the dual variables are reduced to the set $X^* \times C^- \setminus \{0\}$, while in [30], the dual space is the set of all proper conaffine functions including $z^* = 0$. In [18] the dual space is the set of all conaffine functions prohibiting $z^* = 0$. There, improper scalarizations play an important role while in the present approach we avoid those at the expense of including $z^* = 0$ in the dual space.

The following general result is partly due to F. Heyde.

Lemma 6.4 *Let $s : W \to W$ be a bijective duality on a* inf*–residuated order complete set W with neutral element θ, then the following two statements are equivalent:*

$$\begin{aligned} \forall v, w \in W : \quad & s\left(s^{-1}(w) + v\right) = w \dot{-} v \\ \forall w \in W : \quad & s\left(w\right) = s\left(\theta\right) \dot{-} w; \quad s(w) = s^{-1}\left(w\right). \end{aligned} \tag{6.6}$$

PROOF. Indeed, let $s\left(s^{-1}(w) + v\right) = w \dot{-} v$ be fulfilled for all $w, v \in W$, and set $w_0 = s\left(\theta\right)$, then

$$s\left(v\right) = s\left(s^{-1}\left(w_0\right) + v\right) = s\left(\theta\right) \dot{-} v.$$

Moreover,

$$s\left(\theta\right) \leq \left(s\left(\theta\right) \dot{-} v\right) + v,$$

thus $s\left(s\left(v\right)\right) \leq v$ holds true for all $v \in W$. As $s : W \to W$ is a duality, this is equivalent to

$$\forall v \in W : \quad s^{-1}(v) \leq s\left(v\right),$$

and setting $w = s^{-1}(v)$ this is $w \leq s\left(s\left(w\right)\right)$ for all $w \in W$, as s is bijective. Thus the first implication is proven. On the other hand assume $s(w) = s^{-1}(w)$ and $s\left(w\right) = s\left(\theta\right) \dot{-} w$ for all $w \in W$. By the residuation property,for all $v, w \in W$ holds

$$\begin{aligned} s\left(s^{-1}\left(w\right) + v\right) &= s\left(\theta\right) \dot{-} \left(\left(s\left(\theta\right) \dot{-} w\right) + v\right) \\ &= \left(s\left(\theta\right) \dot{-} \left(s\left(\theta\right) \dot{-} w\right)\right) \dot{-} v \\ &= s\left(s\left(w\right)\right) \dot{-} v \\ &= w \dot{-} v, \end{aligned}$$

using the well known property

$$\forall u, v, w \in W : \quad u \dot{-} \left(v + w\right) = \left(u \dot{-} v\right) \dot{-} w,$$

compare [3, Theorem 11.3]. $\square$

If $g : X \to \mathcal{P}^{\triangle}$ is convex and closed, then each image $g(x)$ is a closed convex set, $g(x) \in Q^{\triangle}$, compare Remark 4.3. In general, there does not exist a bijective duality on $Q^{\triangle}$ satisfying the Property (6.6). Thus, the conjugate function $c(g) = g^*$ as a mapping from $(Q^{\triangle})^X$ into $(Q^{\triangle})^{X^* \times C^-}$ in general is not a $(*, s)$–duality. The situation is different for functions with the image space $\mathcal{P}^{\triangle}$.

Proposition 6.5 *Let $s : \mathcal{P}^\triangle \to \mathcal{P}^\triangle$ be a bijection on $\mathcal{P}^\triangle$, defined by $s(A) = (Z \setminus -A)$ for all $A \in \mathcal{P}^\triangle$. Then $A +^s B = A \dot{-} B$ is satisfied for all $A, B \in \mathcal{P}^\triangle$. The mapping $\mathrm{c} : (\mathcal{P}^\triangle)^X \to (\mathcal{P}^\triangle)^{X^* \times C^-}$ with*

$$\forall g : X \to \mathcal{P}^\triangle : \quad \mathrm{c}\,(g) = g^*$$

is a $(+, s)$–duality in the sense of [11, Definition 4.3], compare Formula (2.8). The associated coupling function is

$$(x, (x^*, z^*)) \mapsto S_{(x^*,z^*)}(x).$$

The bidual in the sense of (2.9) is

$$g^{**}(x) = \bigcap_{(x^*,z^*)\in X^*\times C^-} \left(S_{(x^*,z^*)}(x) \dot{-}\, g^*(x^*, z^*) \right).$$

Proof. Indeed, $s(A) = s^{-1}(A)$ is valid for all $A \in \mathcal{P}^\triangle$. By definition,

$$A +^s B = Z \setminus -(B + Z \setminus -A)$$

and $z \in -(B + Z \setminus -A)$ is satisfied if and only if

$$\exists \bar{a} \in Z \setminus -A, \exists b \in B : \quad z = -\bar{a} + (-1)b,$$

or equivalently

$$-(B + Z \setminus -A) = \{z \in Z \mid \exists b \in B : \, z + b \in -(Z \setminus -A)\}.$$

By $-(Z \setminus -A) = Z \setminus A$, we can conclude

$$A +^s B = Z \setminus \{z \in Z \mid \exists b \in B : \, z + b \in (Z \setminus A)\},$$

thus $A +^s B = A \dot{-} B$. The defining Equalities (2.5) and (2.6) are easily checked.

□

Definition 6.6 *To a function $g : X \to \mathcal{P}^\triangle$, the biconjugate $g^{**} : X \to \mathcal{P}^\triangle$ in $x \in X$ is defined by*

$$g^{**}(x) = \bigcap_{(x^*,z^*)\in X^*\times C^-} \left(S_{(x^*,z^*)}(x) \dot{-}\, g^*(x^*, z^*) \right). \tag{6.7}$$

Theorem 6.7 (Biconjugation Theorem) *Let $g : X \to \mathcal{P}^\triangle$, $x \in X$ then*

$$(\mathrm{cl}\,\mathrm{co}\, g)(x) = g^{**}(x). \tag{6.8}$$

Moreover,

$$g^{**}(x) = \bigcap_{z^*\in C^-} \{z \in Z \mid (\varphi_{g,z^*})^{**}(x) \le -z^*(z)\}.$$

The function $\mathrm{cl}\,\mathrm{co}\, g$ is proper or constant $\emptyset$ or Z, if and only if equality is satisfied when omitting $z^* = 0$:

$$g^{**}(x) = \bigcap_{z^*\in C^-\setminus\{0\}} \{z \in Z \mid (\varphi_{g,z^*})^{**}(x) \le -z^*(z)\}. \tag{6.9}$$

PROOF. By Proposition 6.2 the conjugate of a function is represented by $g^*(x^*, z^*) = S_{(\varphi^*_{g,z^*}, z^*)}(x^*)$ for all $(x^*, z^*) \in X^* \times C^-$, thus by Definition 6.6 we conclude

$$g^{**}(x) = \bigcap_{z^* \in C^-} \left(\bigcap_{x^* \in X^*} \left(S_{(x^*,z^*)}(x) \dot{-} S_{(\varphi^*_{g,z^*}, z^*)}(x^*) \right) \right)$$

for all $x \in X$. By Proposition 5.6,

$$S_{(x^*,z^*)}(x) \dot{-} S_{(\varphi^*_{g,z^*}, z^*)}(x^*) = \left\{ z \in Z \mid x^*(x) \dot{-} \varphi^*_{g,z^*}(x^*) \leq -z^*(z) \right\}$$

is fulfilled for all $x \in X$ and all $(x^*, z^*) \in X^* \times C^-$, thus from Proposition 5.6 we get

$$\forall x \in X : \quad g^{**}(x) = \bigcap_{z^* \in C^-} \left\{ z \in Z \mid \varphi^{**}_{g,z^*}(x) \leq -z^*(z) \right\}.$$

By the scalar biconjugation theorem, $(\operatorname{cl}\operatorname{co}\varphi_{g,z^*})(x) = \varphi^{**}_{g,z^*}(x)$ is met for all $x \in \operatorname{dom} g$. Moreover, $\operatorname{cl}\operatorname{co}\varphi_{g,0} = \varphi^{**}_{g,0}$ holds true, as $\operatorname{cl}\operatorname{co}\varphi_{g,0}$ is either proper or constant $+\infty$, thus

$$\begin{aligned}
\forall x \in \operatorname{dom}(\operatorname{cl}\operatorname{co} g) : \quad & g^{**}(x) = (\operatorname{cl}\operatorname{co} g)(x); \\
\forall x \notin \operatorname{dom}(\operatorname{cl}\operatorname{co} g) : \quad & g^{**}(x) \subseteq \{ z \in Z \mid (\varphi_{g,0})^{**}(x) \leq 0 \} \\
& \qquad\quad = \emptyset,
\end{aligned}$$

proving $(\operatorname{cl}\operatorname{co} g)(x) = g^{**}(x)$ for all $x \in X$ and for all $x \in X$

$$g^{**}(x) = \bigcap_{z^* \in C^-} \{ z \in Z \mid (\varphi_{g,z^*})^{**}(x) \leq -z^*(z) \}.$$

If $\operatorname{cl}\operatorname{co} g$ is proper or constant Z or $\emptyset$, then there exists $z^* \in C^- \setminus \{0\}$ such that $\operatorname{cl}\operatorname{co}\varphi_{g,z^*}(x) = \varphi^{**}_{g,z^*}(x) = +\infty$ for all $x \notin \operatorname{dom}(\operatorname{cl}\operatorname{co} g)$. In this case

$$g^{**}(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \{ z \in Z \mid (\varphi_{g,z^*})^{**}(x) \leq -z^*(z) \}$$

is satisfied for all $x \in X$. Finally, if Equation (6.9) is met and $g^{**}(x) = Z$ for some $x \in X$, then

$$\forall z^* \in C^- \setminus \{0\} : (\varphi_{g,z^*})^{**}(x) = -\infty.$$

In this case, g^{**} is constant Z, proving the statement.

□

Obviously, if $(\operatorname{cl}\operatorname{co} g)$ is z^*–proper (or constant Z or $\emptyset$), then $(\operatorname{cl}\operatorname{co}\varphi_{g,z^*})$ is proper (or constant $-\infty$ or $+\infty$) and thus $\varphi^{**}_{g,z^*}(x) = \operatorname{cl}\operatorname{co}\varphi_{g,z^*}(x)$ holds for all $x \in X$. Especially, as $g^{**}(x) = (\operatorname{cl}\operatorname{co} g)(x)$ was proven in Theorem 6.7, $\varphi^{**}_{g,z^*}(x) = \operatorname{cl}\varphi_{g^{**},z^*}(x)$ is fulfilled for all $x \in X$.

Remark 6.8 *Under additional assumptions on the order cone C such as closedness and pointedness, a conjugate of a vector–valued function $f : X \to Z$ is defined in [4, 31] and the references therein. The pre image space of the conjugate is the set of continuous linear operators $T : X \to Z$, the conjugate is defined by*

$$f^{+}(T) = \sup_{x\in X} \left(T(x) - f(x)\right).$$

To guarantee the existence of $f^{+}(T)$, the order induced by C is assumed to fulfill a least upper bound property [31] or even order completeness [4] is assumed. Identifying $f_C(x) = f(x) \oplus C$, the following representation is fulfilled.

$$\begin{aligned} f^{+}(T) + H(z^*) &= (f_C)^* \left(-T^* z^*, z^*\right), \\ f^{+}(T) + \operatorname{cl} C &= \bigcap_{z^*\in C^-} (f_C)^* \left(-T^* z^*, z^*\right). \end{aligned}$$

Thus, results on the conjugate f^{+} are included in the more general results on our set–valued conjugate.

The reader is referred to [14, Proposition 12, 13; Theorem 2, 3], [29, Section 4], [11, Corollary 4.2] for a more thorough investigation of the dualities $\mathrm{c}\,(g) = g^*$ and $\mathrm{c}\,'(g^*) = g^{**}$.

Theorem 6.7 is a set–valued Fenchel–Moreau Theorem, including the improper case alongside to the proper case. In [18, Theorem 5.33], this theorem has been proven using conaffine functions as dual variables. The proper case can be found in [14, Theorem 2] or in [29, Theorem 4.1.15].

7 Duality results

In analogy to the scalar case, a Chain–Rule as well as a Sandwich Theorem and the Fenchel–Rockafellar Duality Theorem can be proven for set–valued functions. We abbreviate the proofs by citing the known scalar results and applying Proposition 6.2 and Theorem 6.7. Direct proofs for a special case can be found in [14, 15]. There, strong duality results are formulated under the additional assumption of an inner point $(x_0, z_0) \in \operatorname{int} \operatorname{epi} g$, compare Proposition 7.1(b).We will show that continuity of g in x_0 in the sense of [1, 12], too, is a sufficient assumption for strong duality results.

Proposition 7.1 *[19] Let $g : X \to \mathcal{P}^{\triangle}$ be a function, $x \in X$ and $z^* \in C^-$ and let one of the following assumptions be satisfied.*

(a) The function g is convex and upper continuous in $x_0 \in \operatorname{dom} g$ in the sense of [1, Definition 1.4.3.], [12, Definition 2.5.1.] i.e. $D \subseteq Z$ is an open set with $g(x) \subseteq D$, then there exists a 0–neighborhood $V \subseteq X$ such that

$$\forall x \in V : \quad g(x_0 + x) \subseteq D; \tag{7.1}$$

(b) The function g is convex and there is $z_0 \in Z$ such that $(x_0, z_0) \in \operatorname{int} \operatorname{epi} g$.

Then φ_{g,z^*} is convex and either continuous in x_0 or $\varphi_{g,z^*}(x_0 + x) = -\infty$ is satisfied for all elements x of an open subset $V \subseteq X$ with $0 \in V$.

It is easy to check that under the assumptions of Proposition 7.1, $\varphi_{g,0}$ is continuous in x_0 and $\varphi_{g,0}(x_0) = 0$. Under the assumptions of Proposition 7.1, each scalarization satisfies $\varphi_{g,z^*}(x_0) = (\operatorname{cl}\operatorname{co}\varphi_{g,z^*})(x_0)$. Hence, the assumptions of Proposition 7.1 are sufficient for the following two equalities

$$g(x_0) = (\operatorname{cl}\operatorname{co} g)(x_0) = \operatorname{cl}\operatorname{co}\left(g(x_0)\right) \tag{7.2}$$

Definition 7.2 *Let $g_1, g_2 : X \to \mathcal{P}^\triangle$ be two functions, Y be another separated locally convex space, $f : X \to \mathcal{P}^\triangle$ a function, $(x^*, z^*) \in X^* \times C^- \setminus \{0\}$ and $T : X \to Y$ a linear continuous operator.*

(a) The sup*–addition in the set $(\mathcal{P}^\triangle)^{X^* \times C^- \setminus \{0\}}$ is defined by $(g_1^* \underset{\cdot}{+} g_2^*)(x^*, z^*) = Z$ if either $g_1^*(x^*, z^*) = Z$ or $g_2^*(x^*, z^*) = Z$ and else*

$$(g_1^* \underset{\cdot}{+} g_2^*)(x^*, z^*) = g_1^*(x^*, z^*) + g_2^*(x^*, z^*).$$

(b) Define the infimal convolution of g_1^* and g_2^* with respect to $\underset{\cdot}{+}$ by

$$(g_1^* \underset{\cdot}{\square} g_2^*)(x^*, z^*) = \operatorname{cl} \bigcup_{x_1^* + x_2^* = x^*} \left(g_1^*(x_1^*, z^*) \underset{\cdot}{+} g_2^*(x_2^*, z^*)\right)$$

(c) Define

$$(T^* f^*)(x^*, z^*) = \operatorname{cl} \bigcup_{T^* y^* = x^*} f^*(y^*, z^*).$$

Using the duality introduced in Proposition 6.5, a representation analog to the scalar Formula (3.11) can be proven for all $(x^*, z^*) \in X^* \times C^- \setminus \{0\}$.

$$(g_1^* \underset{\cdot}{+} g_2^*)(x^*) = s^{-1}\left(s\left(g_1^*(x^*, z^*)\right) + s\left(g_2^*(x^*, z^*)\right)\right).$$

Applying the scalar Chain Rule 3.6 and Propositions 5.5, 5.6 and 6.2, we get the following result.

Theorem 7.3 (Chain–Rule) *Let $g : X \to \mathcal{P}^\triangle$, $f : Y \to \mathcal{P}^\triangle$ be two functions and $T : X \to Y$, $S : Y \to X$ linear continuous operators, $x^* \in X^*$ and $z^* \in C^- \setminus \{0\}$.*

(a) For $x \in X$ define

$$\left(g \square^{\cdot} Sf\right)(x) = \operatorname{cl} \bigcup_{\bar{x} \in X} \left(g(x - \bar{x}) + \inf_{Sy = \bar{x}} f(y)\right).$$

The following inequality holds true.

$$\left(g \square^{\cdot} Sf\right)^*(x^*, z^*) = (g^* \underset{\cdot}{+} f^* S^*)(x^*, z^*)$$

(b) It holds

$$(g + fT)^*(x^*, z^*) \supseteq (g^* \underset{\cdot}{\square} T^* f^*)(x^*, z^*).$$

(c) *If either g or f is the constant mapping $\emptyset$, then for all $x^* \in X^*$ and all $y^* \in Y^*$ it holds*

$$(g+fT)^*(x^*,z^*) = (g^* \square_{\boldsymbol\cdot} T^*f^*)(x^*,z^*) = Z;$$
$$(g+fT)^*(x^*,z^*) = g^*(x^* - T^*y^*, z^*) +_{\boldsymbol\cdot} f^*(y^*,z^*).$$

(d) *If $(fT)(x_0) + H(z^*) = Z$ for some $x_0 \in \operatorname{dom} g$ or if both f and g are convex and one of the assumptions in Proposition 7.1 is satisfied for f in an element of $T(\operatorname{dom} g)$, then for all $x^* \in X^*$*

$$Z \neq (g+fT)^*(x^*,z^*) = (g^* \square_{\boldsymbol\cdot} T^*f^*)(x^*,z^*)$$

is satisfied and it exists $y^* \in Y^*$ such that

$$(g+fT)^*(x^*,z^*) = g^*(x^* - T^*y^*, z^*) + f^*(y^*,z^*).$$

Proof. By Proposition 6.2, the conjugate of a function $h : X \to \mathcal{P}^\triangle$ can be represented as

$$h^*(x^*,z^*) = \left\{ z \in Z \mid \varphi^*_{h,z^*}(x^*) \le -z^*(z) \right\}.$$

Applying Propositions 5.5, 5.6 and Theorem 3.6 we may conclude for $z^* \in C^- \setminus \{0\}$

$$\begin{aligned}
\left(g \square^{\boldsymbol\cdot} Sf\right)^*(x^*,z^*) &= \left\{ z \in Z \mid \left(\varphi^*_{g,z^*} +_{\boldsymbol\cdot} \varphi^*_{f,z^*} S^* \right)(x^*) \le -z^*(z) \right\} \\
&= \left(g^* +_{\boldsymbol\cdot} f^* S^*\right)(x^*,z^*); \\
\left(g +^{\boldsymbol\cdot} fT\right)^*(x^*,z^*) &= \left\{ z \in Z \mid \left(\varphi_{g,z^*} +^{\boldsymbol\cdot} \varphi_{f,z^*} T \right)^*(x^*) \le -z^*(z) \right\} \\
&\supseteq \left\{ z \in Z \mid \left(\varphi^*_{g,z^*} \square_{\boldsymbol\cdot} T^* \varphi^*_{f,z^*} \right)(x^*) \le -z^*(z) \right\} \\
&= \left(g^* \square_{\boldsymbol\cdot} T^* f^*\right)(x^*,z^*)
\end{aligned}$$

proving (a) and (b). If $g \equiv \emptyset$, then so is $g + fT$ and $g^* = (g+fT)^* \equiv Z$, proving (c).

To prove the rest, notice that

$$(g+fT)^*(x^*,z^*) \supseteq (g^* \square_{\boldsymbol\cdot} T^* f^*)(x^*,z^*) \supseteq \operatorname{cl} \bigcup_{y^* \in Y^*} \left(g^*(x^* - T^*y^*, z^*) + f^*(y^*,z^*) \right).$$

Let one of the assumptions of Proposition 7.1 be satisfied for f in a point Tx_0 with $x_0 \in \operatorname{dom} g$, then everything is proven by Theorem 3.6 and Proposition 6.2 while if $(fT)(x_0) = Z$ with $x_{:}0 \in \operatorname{dom} g$, then the conjugate $(g+fT)^*$ is constant $\emptyset$ and thus equality holds by (b). $\square$

As in the scalar case, equality in Theorem 7.3 (a) and (c) does not hold true with the usual Minkowsky (inf–) addition on the right hand side. Notice however, that here as well as in the scalar case (see Theorem 3.6 (d)) we do not assume properness for the strong Chain–Rule in Theorem 7.3 (d). Setting $g = 0$ or $X = Y$ and $S = T = id$, a sum–rule and a multiplication–rule are immediate corollaries of Theorem 7.3.

An analog of Theorem 7.3 can be proven for $z^* = 0$, too. However, the preimage space of the conjugate needs to be extended to the set of all conaffine functions, compare [30].

Theorem 7.4 (Fenchel-Rockafellar-Duality) *Let Y be a locally convex separated space with topological dual Y^*. To $g : X \to \mathcal{P}^\triangle$, $f : Y \to \mathcal{P}^\triangle$ and a linear continuous operator $T : X \to Y$ and $z^* \in C^- \setminus \{0\}$, denote*

$$P = \operatorname{cl} \operatorname{co} \bigcup_{x \in X} \left(g(x) + f(Tx) \right); \tag{7.3}$$

$$D(z^*) = \bigcap_{y^* \in Y^*} H(z^*) -^{\boldsymbol\cdot} \left(g^*(T^*y^*, z^*) + f^*(-y^*, z^*) \right). \tag{7.4}$$

(a) It holds $D(z^*) \supseteq P$, the weak duality.

(b) If for f one of the assumptions in Proposition 7.1 is in force for an element in $T(\operatorname{dom} g)$, then $\operatorname{cl}(P + H(z^*)) = D(z^*) \neq \emptyset$ holds and it exists $y^*_{z^*} \in Y^*$ such that

$$\operatorname{cl}(P + H(z^*)) = H(z^*) \dot{-} \left(g^*(T^* y^*_{z^*}, z^*) + f^*(-y^*_{z^*}, z^*)\right) \neq \emptyset.$$

In this case, or if $fT(x_0) = Z$ for some $x_0 \in \operatorname{dom} g$,

$$P = \bigcap_{z^* \in C^- \setminus \{0\}} D(z^*)$$

holds true and it exists a set $\{y^*_{z^*} \in Y^* \mid z^* \in C^- \setminus \{0\}\}$ such that

$$P = \bigcap_{z^* \in C^- \setminus \{0\}} H(z^*) \dot{-} \left(g^*(T^* y^*_{z^*}, z^*) + f^*(-y^*_{z^*}, z^*)\right).$$

Proof. If $z^* \in C^- \setminus \{0\}$, then the following inequality is met

$$\sup_{y^* \in Y^*} \left(0 \dot{-} \left(\varphi^*_{g,z^*}(T^* y^*) \dot{+} \varphi^*_{f,z^*}(-y^*)\right)\right) \leq \inf_{x \in X} \left(\varphi_{g,z^*}(x) + \varphi_{f,z^*} T(x)\right)$$

and equality holds, if φ_{f,z^*} and φ_{g,z^*} are proper functions and φ_{f,z^*} is continuous in $Tx \in Y$ with $x \in \operatorname{dom} \varphi_{g,z^*}$ or if either scalarization attains the value $-\infty$ within the domain of the other. Applying Propositions 5.5, 5.6 and 6.2 proves the statement. □

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